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# Multipliers with Respect to Spectral Measures in Banach Spaces and Approximation. II. One-Dimensional Fourier Multipliers*

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In our previous studies (cf. Part I and the literature referred to there; also [28]) we derived a criterion for radial multipliers in connection with Riesz-bounded spectral measures. The present paper is concerned with extensions of the theory covering multipliers which are not necessarily radial. We shall restrict the discussion to one-dimensional Fourier transforms (and coefficients). Nevertheless, it would be possible to present these extensions in the general Banach space-frame of Part I. Indeed, the procedure will be quite the same, exploiting the fact that the family of (Riesz means-) operators $\{(R, \alpha)_\rho\}_{\rho>0}$ (cf. Part I (3.1); this paper (6.1)) may be considered as a test set for certain multiplier criteria.

## 6.1. MULTIPLIER CRITERIA

The situation is that of Section 5.2 for $n = 1$, i.e., we deal with functions $f \in L^p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, and their Riesz means of order $\alpha$

$$(R, \alpha)_\rho f(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} f(x-u)\, \rho b_\alpha(\rho u)\, du \equiv \rho b_\alpha(\rho\, \cdot) * f(x), \tag{6.1}$$

* This paper is a sequel to one with the same title published in this journal Vol. 8, pp. 335–356. The contents (and notations) of the first part are assumed to be known. References are in alphabetical order in each paper; they, as well as the sections, are numbered consecutively throughout this series. The contribution of W. Trebels was supported by DFG-Grant Bu 166/23.

where $b_\alpha$ is given by its (distributional) Fourier transform

$$b_\alpha{}^\wedge(v) = \begin{cases} (1-|v|)^\alpha, & |v| \leq 1 \\ 0, & |v| \geq 1 \end{cases}, \quad \phi^\wedge(v) = (2\pi)^{-1} \int_{-\infty}^{\infty} \phi(u) e^{-ivu}\, du.$$

(Note that in comparison with (3.1) we change the notation and write $r_\alpha = b_\alpha{}^\wedge$.) Then it is well known that $b_\alpha \in L^1(\mathbb{R})$ for each $\alpha > 0$. From this we shall derive sufficient criteria for functions $\lambda(v)$ on $\mathbb{R}$ to belong to $[L^1(\mathbb{R})]^\wedge$, the set of $L^1$-transforms.

In fact, the case $\lambda$ being an even function, i.e., $\lambda(v) = \lambda(|v|)$, is already covered by our previous results.

THEOREM 6.1. *Let $\lambda(v) \in C_0(\mathbb{R})$ be even. If*[1] $\lambda \in BV_{\alpha+1}$ *(cf. Part I, (3.3), this paper (6.3)) for some* $\alpha > 0$, *then* $\lambda(v) \in [L^1(\mathbb{R})]^\wedge$.

Next we consider odd functions $\lambda$, i.e., $\lambda(v) = \{\text{sgn}\, v\}\, \lambda(|v|)$, where $\text{sgn}\, v = -1$ for $v < 0$, $=0$ for $v = 0$, and $=1$ for $v > 0$. Here we make use of the following property of the Riesz kernel.

LEMMA 6.1. *Setting* $\{\text{sgn}\, v\}|v|^\beta b_\alpha{}^\wedge(v) = b_{\alpha,\beta}^\wedge(v)$ *one has* $b_{\alpha,\beta} \in L^1(\mathbb{R})$ *for each* $\alpha > 0$, $\beta > 0$.

*Proof.* By Theorem 6.1 $|v|^2 b_\alpha{}^\wedge(v) = [-b_\alpha'']^\wedge(v) \in [L^1]^\wedge$. Therefore,

$$\| b_\alpha(x+3t) - 3b_\alpha(x+t) + 3b_\alpha(x-t) - b_\alpha(x-3t)\|_1 \leq 12t^2 \| b_\alpha''\|_1 .$$

For some fixed $0 < \beta < 2$ this implies that

$$f_{\epsilon,\beta}(x) = \int_\epsilon^\infty t^{-1-\beta}[b_\alpha(x+3t) - 3b_\alpha(x+t) + 3b_\alpha(x-t) - b_\alpha(x-3t)]\, dt,$$

is Cauchy for $\epsilon \to 0+$. Hence there exists $f_\beta \in L^1$ such that

$$\lim_{\epsilon\to 0+} \| f_{\epsilon,\beta} - f_\beta\|_1 = 0.$$

However,

$$f_{\epsilon,\beta}^\wedge(v) = \int_\epsilon^\infty t^{-1-\beta}(e^{ivt} - e^{-ivt})^3\, dt \cdot b_\alpha{}^\wedge(v),$$

which implies $f_\beta{}^\wedge(v) = C_\beta\{-i\, \text{sgn}\, v\}|v|^\beta b_\alpha{}^\wedge(v)$ with some constant $C_\beta$. Now the proof for arbitrary $\beta > 0$ is immediate. Indeed, with $C^\infty(\mathbb{R})$ the set of infinitely differentiable functions on $\mathbb{R}$, let (cf. Part I, Section 4.2)

$$\chi \in C^\infty(\mathbb{R}) \quad \text{even with } \chi(v) = 1 \quad \text{for} \quad |v| \leq 1, \quad =0 \text{ for } \quad |v| \geq 2. \qquad (6.2)$$

[1] Of course, here and in the following the restriction of $\lambda$ to $(0, \infty)$ is meant in connection with the moment spaces $BV_{\alpha+1}^{\omega,\sigma}$.

Then $|v|^\gamma \chi(v) \in [L^1]^\wedge$ for each $\gamma > 0$ by Theorem 6.1, and the proof is complete.

Let us observe that $b_{\alpha,\beta}(x) = i(D^{(\beta)} b_\alpha)^\sim(x)$, the Hilbert transform of the $\beta$th Riesz derivative of $b_\alpha$ (cf. [27, Chapter XI]).

In connection with odd functions it is appropriate to consider the following modification of the class $BV_{\alpha+1}$ (cf. Part I (3.3)) for $\alpha, \omega, \sigma \geqslant 0$

$$BV_{\alpha+1}^{\omega,\sigma} = \{\lambda(t) \in C_0(0, \infty) : t^{-\sigma}\lambda(t) \in BV_{\alpha+1}^{\omega}\}, \tag{6.3}$$

with norm $\|\lambda\|_{BV_{\alpha+1}^{\omega,\sigma}} = \|t^{-\sigma}\lambda(t)\|_{BV_{\alpha+1}^{\omega}}$, where (cf. [28])

$$BV_{\alpha+1}^{\omega} = \Big\{\lambda(t) \in C_0(0, \infty);\ \lambda^{(\gamma)}, \ldots, \lambda^{(\alpha-1)} \in AC_{\mathrm{loc}}(0, \infty),\ \lambda^{(\alpha)} \in BV_{\mathrm{loc}}(0, \infty),$$

$$\|\lambda\|_{BV_{\alpha+1}^{\omega}} = \frac{1}{\Gamma(\alpha + 1 + \omega)} \int_0^\infty t^{\alpha+\omega} |d\lambda^{(\alpha)}(t)| < \infty\Big\}. \tag{6.4}$$

For the notations see Section 3 (Part I), in particular, for $\gamma = \alpha - [\alpha]$, $[\alpha]$ being the largest integer $\leqslant \alpha$, and $C_0(0, \infty)$, the set of continuous functions $\lambda(t)$ on $(0, \infty)$ with $\lim_{t\to\infty} \lambda(t) = 0$. Note that $BV_{\alpha+1}^{\omega} \subset BV_{\beta+1}^{\omega}$ in the sense of continuous embedding for all $0 \leqslant \beta \leqslant \alpha$, $\omega \geqslant 0$, and that for any $\lambda \in BV_{\alpha+1}^{\omega}$, $\alpha, \omega \geqslant 0$, one has

$$\lambda(s) = \frac{(-1)^{[\alpha]+1}}{\Gamma(\alpha + 1)} \int_s^\infty (t - s)^\alpha \, d\lambda^{(\alpha)}(t), \tag{6.5}$$

(cf. [28], also Part I (3.5)). Moreover, $\lambda \in BV_{\alpha+1}^{\omega,\sigma}$ implies

$$\|\lambda(t/\rho)\|_{BV_{\alpha+1}^{\omega,\sigma}} = \rho^{\omega-\sigma} \|\lambda(t)\|_{BV_{\alpha+1}^{\omega,\sigma}}, \qquad (\rho > 0), \tag{6.6}$$

so that $BV_{\alpha+1}^{\sigma;\sigma}$-norms are invariant with respect to dilations. This is important in connection with uniform bounds for multipliers of Fejér's type (i.e., $\lambda_\rho(t) = \lambda(t/\rho)$).

THEOREM 6.2. *Let $\lambda(v) \in C_0(\mathbb{R})$ be odd. If $\lambda \in BV_{\alpha+1}^{\sigma,\sigma}$ for some $\alpha > 0$, $\sigma > 0$, then $\lambda(v) \in [L^1(\mathbb{R})]^\wedge$.*

*Proof.* Setting $\lambda_\sigma(t) = t^{-\sigma}\lambda(t)$ and

$$g(x) = \frac{(-1)^{[\alpha]+1}}{\Gamma(\alpha + 1)} \int_0^\infty t^{\alpha+\sigma}[tb_{\alpha,\sigma}(tx)] \, d\lambda_\sigma^{(\alpha)}(t),$$

then $g \in L^1$ by Lemma 6.1; in fact, $\|g\|_1 \leqslant A_{\alpha,\sigma} \|b_{\alpha,\sigma}\|_1 \|\lambda\|_{BV_{\alpha+1}^{\sigma,\sigma}}$.

Moreover, by Fubini's theorem (cf. this paper (6.5))

$$g^\wedge(v) = \frac{(-1)^{[\alpha]+1}}{\Gamma(\alpha+1)} \int_0^\infty t^{\alpha+\sigma} b^\wedge_{\alpha,\sigma}(v/t)\, d\lambda_\sigma^{(\alpha)}(t)$$

$$= \{\operatorname{sgn} v\}\, |v|^\sigma \frac{(-1)^{[\alpha]+1}}{\Gamma(\alpha+1)} \int_{|v|}^\infty t^\alpha \left(1 - \frac{|v|}{t}\right)^\alpha d\lambda_\sigma^{(\alpha)}(t)$$

$$= \{\operatorname{sgn} v\}\, |v|^\sigma \lambda_\sigma(|v|) = \{\operatorname{sgn} v\}\, \lambda(|v|) = \lambda(v),$$

which proves the theorem.

Let us remark that if $\lambda \in [L^1]^\wedge$ is odd, then necessarily $|\int_a^b t^{-1}\lambda(t)\, dt| \leq A$ uniformly for $0 < a < b < \infty$ (cf. [29, p. 8; 31, p. 31]); for a converse result compare [32, p. 185] in the trigonometric case.

Concerning arbitrary functions $\lambda$, we recall that each $\lambda(v)$ on $\mathbb{R}$ may be decomposed into an even part $\lambda_1$ and odd part $\lambda_2$ via

$$\lambda(v) = \frac{\lambda(v) + \lambda(-v)}{2} + \frac{\lambda(v) - \lambda(-v)}{2} = \lambda_1(v) + \lambda_2(v). \qquad (6.7)$$

Thus one has as an immediate consequence.

THEOREM 6.3. *Let $\lambda(v) \in C_0(\mathbb{R})$ be an arbitrary function on $\mathbb{R}$, and let $\lambda_1$ and $\lambda_2$ be its even and odd part, respectively. If $\lambda_1 \in BV_{\alpha+1}$ and $\lambda_2 \in BV_{\alpha+1}^{\sigma,\sigma}$ for some $\alpha > 0$, $\sigma > 0$, then $\lambda(v) \in [L^1(\mathbb{R})]^\wedge$.*

Let us observe that, apart from the usual differentiability properties, a sufficient condition for $\lambda_1$ (cf. (6.7)) to belong to $BV_{\alpha+1}$ is given by $\int_{-\infty}^\infty |v|^\alpha |d\lambda^{(\alpha)}(v)| < \infty$ where for $v < 0$, $0 < \alpha < 1$ (cf. Part I, Section 3)

$$\lambda^{(\alpha)}(v) = \lim_{a\to-\infty} \frac{d}{dv}\left[\frac{1}{\Gamma(1-\alpha)} \int_a^v (v-u)^{-\alpha} \lambda(u)\, du\right].$$

Concerning the condition $\lambda_2 \in BV_{\alpha+1}^{\sigma,\sigma}$, it is sufficient to show that ($a > 0$, $0 < \alpha < \delta < 1$, $0 < s < \frac{1}{4}$)

$$\int_{|v|>a} |v|^{1+\sigma} |[|v|^{-\sigma} \lambda(v)]''|\, dv < \infty,$$

$$\int_0^a t^{\alpha+\sigma} \left| \left[\frac{\lambda(t+s) - \lambda(-t-s)}{(t+s)^\sigma} - \frac{\lambda(t) - \lambda(-t)}{t^\sigma}\right]' \right| dt = O(s^\delta).$$

In this connection one may compare the foregoing with the following result (in case $n = 1$) of Löfström [30] and Boman [26]. If $\lambda(v)$, defined and continuously differentiable on $\mathbb{R} - \{0\}$, has compact support, and if there exist constants $C$ and $\gamma > 0$ such that

$$|(d/dv)^k \lambda(v)| \leqslant C\,|v|^{\gamma-k} \qquad (v \in \mathbb{R} - \{0\}, k = 0, 1),$$

then $\lambda \in [L^1]^\wedge$. Now, let $\epsilon > 0$ and consider the function

$$\lambda(v) = \{\operatorname{sgn} v\} \log^{-1-\epsilon}(1/|v|)\, \chi(3v),$$

where $\chi$ is given by (6.2). Then $\lambda$ is a continuous function with $\lim_{v\to 0} \lambda(v) = 0$. Since $\lambda$ does not satisfy a Lipschitz condition of any positive order (indeed, $\lim_{t\to 0+} t^{-\gamma}\lambda(t) = \infty$ for any $\gamma > 0$), the criterion of Boman–Löfström does not work. However, $\lambda \in BV_2^{\sigma,\epsilon}$ for any $\sigma > 0$ so that Theorem 6.2 delivers $\lambda(v) \in [L^1]^\wedge$.

The proof of Theorem 6.2 is arranged in such a way that a formulation in the general Banach space-frame of Section 2 (Part I) is immediate. Indeed, let $E$ be a spectral measure for the Hilbert space $H$ on $\mathbb{R}$, and let $X$ be a Banach space such that $H \cap X$ is dense in $H$ and $X$. If $\lambda$ is an odd function on $\mathbb{R}$ and the operators (cf. Lemma 6.1)

$$(R, \alpha, \beta)_\rho = \int_{-\infty}^{\infty} r_{\alpha,\beta}(v/\rho)\, dE(v), \qquad r_{\alpha,\beta} = h^\wedge_{\alpha,\beta},$$

are uniformly bounded on $X$ for some $\alpha, \beta > 0$, then $\lambda \in BV_{\alpha+1}^{\beta,\beta}$ implies $\lambda(v) \in M$.

### 6.2. Bernstein-Type Inequalities

In the terminology (for $n = 1$) of Section 5.2 (Part I), let $B_{a,p}$ be the set of entire functions $f(z)$ of exponential type such that $|f(z)| \leqslant Ae^{a|z|}$ and $f(x + i0) \in L^p(\mathbb{R})$ (in case $p = \infty$ we restrict to functions of class $C_0(\mathbb{R})$). With $\chi$ as given by (6.2) consider for some $\alpha > 0$

$$(iv)^\alpha \chi(v) = \cos(\pi\alpha/2)|v|^\alpha \chi(v) + \sin(\pi\alpha/2)\{i \operatorname{sgn} v\}|v|^\alpha \chi(v) = I_1 + I_2, \tag{6.8}$$

say. Then there exists $g \in L^1$ such that $g^\wedge(v) = (iv)^\alpha \chi(v)$. Indeed, $I_1 \in [L^1]^\wedge$ by Theorem 6.1, whereas

$$\int_0^2 t^{1+(\alpha/2)}\, |(d/dt)^2 (t^{\alpha/2}\chi(t))|\, dt < \infty,$$

implies $t^{\alpha/2}\chi(t) \in BV_2^{\alpha/2}$ and thus $t^\alpha\chi(t) \in BV_2^{\alpha/2,\alpha/2}$ so that $I_2 \in [L^1]^\wedge$ by

Theorem 6.2. Therefore, since $f \in B_{a,p} \cap B_{a,2}$ implies $f^\wedge(v) = 0$ for $|v| > a$ by the Paley–Wiener theorem, it follows that

$$\| \mathfrak{F}^{-1} [(iv)^\alpha f^\wedge(v)] \|_p = a^\alpha \left\| \int_{|v| \le a} \left( \frac{iv}{a} \right)^\alpha \chi\left(\frac{v}{a}\right) f^\wedge(v) e^{ixv} \, dv \right\|_p$$

$$= a^\alpha \| ag(a\,\cdot) * f \|_p \le a^\alpha \|g\|_1 \|f\|_p ,$$

for any $f \in B_{a,p} \cap B_{a,2}$. The extension to all of $B_{a,p}$ may be performed analogously to the procedure described in Section 5.2 (Part I) by using a decomposition corresponding to (6.8) and Theorems 6.1 and 6.2. Therefore, interpreting Fourier transforms in the distributional sense (cf. [26]).

COROLLARY 6.1. *For $\alpha > 0$ one has the Bernstein-type inequality*

$$\| \mathfrak{F}^{-1}[(iv)^\alpha f^\wedge(v)] \|_p \le \text{const}\, a^\alpha \|f\|_p \qquad (f \in B_{\alpha,p}).$$

In the standard manner this may be used to derive the classical Bernstein inequality concerning trigonometric polynomials in, e.g., $C_{2\pi}$, the space of continuous, $2\pi$-periodic functions. Indeed, setting (with $g$ as given above)

$$g_n^*(x) = \sum_{k=-\infty}^{\infty} ng(n(x + 2k\pi)), \; [g_n^*]^\wedge(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_n^*(u) e^{-iku} \, du,$$

one has $\| g_n^* \|_{L^1_{2\pi}} \le \|g\|_1$ uniformly for $n \in \mathbb{N}$ and $[g_n^*]^\wedge(k) = g^\wedge(k/n)$ (cf. [27, p. 201]). Thus for any trigonometric polynomial $t_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$ and $\alpha > 0$

$$\left\| \sum_{k=-n}^{n} (ik)^\alpha c_k e^{ikx} \right\|_{C_{2\pi}} \le \|g\|_1 \cdot n^\alpha \|t_n\|_{C_{2\pi}} .$$

In [11] we considered $BV_{\alpha+1}$-spaces, and thus obtained this inequality only for $\alpha = 2s$, $s \in \mathbb{N}$. By the slight extension to $BV_{\alpha+1}^{\sigma,\sigma}$-spaces, however, the above classical inequality is regained for any $\alpha > 0$.

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